Zeta function regularization for a scalar field in a compact domain

G. Ortenzi† and M. Spreafico‡

† Dipartimento di Matematica ed Applicazioni, Università Milano Bicocca, Italy, giolev@matapp.unimib.it

‡ ICMC Universidade São Paulo, São Carlos, Brazil, mauros@icmc.usp.br

Abstract. We express the zeta function associated to the Laplacian operator on $S_r^1 \times M$ in terms of the zeta function associated to the Laplacian on M, where M is a compact connected Riemannian manifold. This gives formulas for the partition function of the associated physical model at low and high temperature for any compact domain M. Furthermore, we provide an exact formula for the zeta function at any value of r when M is a D-dimensional box or a D-dimensional torus; this allows a rigorous calculation of the zeta invariants and the analysis of the main thermodynamic functions associated to the physical models at finite temperature.

1. Introduction

The zeta function regularization procedure is one of the most elegant and rigorous methods to deal with path integrals in quantum field theory. A great effort has been done in the last years to produce exact calculations in different cases of interest [13] [10] [11] [12] [3] with particular attention to exact calculation of the heat kernel coefficients [4]. Notice also some rigorous calculations appeared in the recent mathematical literature: [7] [9] [20] [28] [29] [18]. The purpose of this note is to show the result of a rigorous application of the zeta function regularization technique. In particular, we analyze the basic example considered by Hawking in [16]: a scalar field in a compact domain of the product space time $S_{\beta/2\pi}^1 \times M$ at temperature $T = \frac{1}{\beta}$, and provide a complete treatment of it. Our main motivation is that we can perform all the calculations in a rigorous way and determinate all the quantities of interest without introducing any approximation. Our main result is an effective formula expressing the partition function at finite temperature in terms of the geometry of the spatial background. This allows on one side to describe the behavior of the physical model at low and high temperature for any spatial domain (cfr with [8], where a similar analysis was performed for $M = H^3/\Gamma$); on the other side, to provide explicit formulas that describe the model for some particular geometries, namely the D-dimensional box and the D-dimensional torus, at any value

of the temperature. Our technique is likely to be generalized to other situations, and there are works in progress in various directions.

The partition function for a scalar field ϕ with action $I[\phi] = -\int \phi(x)A\phi(x)d(vol(x))$ in the Euclidean space time and where A is a real elliptic self adjoint second order differential operator, can be formally described by the Feynman path integral $Z = \int \mathcal{D}\phi e^{\rho I[\phi]} = [\det(\rho A)]^{-\frac{1}{2}}$ where ρ is some renormalization constant with dimension of mass or inverse length introduced by hand in order to obtain a physically consistent model [16]‡. With periodic boundary condition in the imaginary time with period β, the background geometry of the flat space time is described by the product space $S^1_{\beta/2\pi} \times M$, with product metric $1 \oplus g$, where (M, g) is a complete Riemannian manifold of dimension D; in particular, we assume M to be compact connected. Following Hawking [16], the path integral can be given a rigorous interpretation in terms of some zeta invariants associated to the underlying geometry. We introduce the zeta function associated to the operator A, $\zeta(s; A) = \sum_{\lambda \in \mathrm{Sp}_0 A} \lambda^{-s}$, (where $\mathrm{Sp}_0 A$ denotes the non vanishing part of the spectrum of A); then, defining the regularized determinant of A [24] [1] by $\log \det A = -\frac{d}{ds}\zeta(s; A)\big|_{s=0}$, we get $\log Z = \frac{1}{2}\zeta'(0; A) - \frac{1}{2}\log \rho \zeta(0; A)$. Under the identification $T = \frac{1}{\beta}$, the partition function for the quantum theoretical model corresponds to the one for a canonical ensemble at temperature T. We will assume this point of view, and we will work out the partition function Z_T for a statistical system calculating the zeta invariants of the underlying geometry. This will allow us to introduce and analyze other interesting thermodynamic functions and to get useful information on the system described at finite temperature T. The main feature of this approach is that we can describe completely the partition function for any theory in the product space time at any value of the temperature, in terms of a zeta function at null temperature (that we will call geometric zeta function) depending only on the geometry of the space, namely of the background physical domain. To get this result, after introducing the zeta function associated to the model in the proper Euclidean setting, we use the particular form of the spectrum to decompose the zeta function. We easy get a function of the temperature T, smooth for all positive T, but the hard point is to get the right analytic extension at $s = 0$, in order to obtain the partition function. In particular, the analytic continuation is not uniform in T for small T, and hence the calculation of $\zeta'(0)$ must be performed at positive T. We are able to get this result, and we can do it uniformly in T for T in any closed interval of the real positive axis (Proposition 3 of Section 2). Moreover, the result is effective, and we can deduce from it both the behaviors for low and high temperature.

Studying the models at low temperature, we show that periodic boundary conditions on the spatial domain give an anomalous behavior for the zero rest mass scalar field that can be corrected by adding a non zero mass term, while studying the behavior of the pressure of the radiation as a function of the volume at different fixed temperatures, we prove the existence of a minimal non vanishing value of the volume below which

‡ Beside in this work we will not treat renormalization problems, we will give effective formulas for Z for any ρ.

the force becomes attractive, as expected because of the Casimir effect [2] [22]. More precisely, the expansions at low temperature of the thermodynamic functions show the following effect of the boundary conditions on the physics of the model: the presence of a spatial zero mode produces an anomalous behavior of the entropy and the specific heat at low temperature. In particular, on a closed (compact, connected, with no boundary) geometry, we always have a zero mode for the Laplacian, due to the constant eigenfunction, and this produces a $\log T$ term in the logarithm of the partition function, and hence a logarithmic divergence in the entropy and a non zero specific heat at low T. Such a term disappears when a boundary is present and opportune boundary conditions are assumed. Furthermore, the presence of a non zero mass term cancels the logarithmic term and recasts a correct behavior for small T on both the domains, independently from the boundary conditions. This is in agreement with the appearance of a logarithmic divergence (infrared divergence) for a massless field classically (see for example [23]). Particularly meaningful in this context is the case $D = 1$, whose associated zeta function appears as the zeta function for the Laplacian with a constant potential on a cylinder or on a torus. Its s-expansion near $s = 0$ is well known since Eisenstein and Kronecker [32], and relates to the theory of modular functions and L-series. The coefficient of the linear term can be expressed using the Dedekind eta function, and using the modular property of the last we can easily relate the behavior at low and high temperature of the partition function of the physical model. It is clear why the presence of a mass term cancels the singular behavior at low T: in fact, such a non homogeneous term breaks the modular property of the eta function. This will be discussed at the end of part 4.3. The D dimensional version of the eta function appearing in the corollary of Proposition 3, has not been studied yet. It does not exhibit any periodicity, but possible modular properties should be very important. A deeper analysis of such function should be very interesting both under a theoretical and applicative point of view.

The study of the thermodynamic function at finite temperature allows to analyze other interesting phenomena. In particular, we investigate the pressure of the radiation at finite temperature as a function of the volume, and we show the existence, for any fixed temperature, of a critical volume V_0 where the pressure of the radiation changes sign, becoming attractive. The analysis performed in part 4.5 of Section 4 shows that below V_0 the pressure is attractive and decreases like $V^{-\frac{D+1}{D}}$, as expected by the Casimir effect [2] [22]. We also show that, up to renormalization, the critical volume depends on the temperature by the law $T_0 V_0^{\frac{1}{D}} = \text{const}$, and the boundary does not affect this effect.

The work is organized as follows. In Section 2 we briefly recall some basic information about the geometry of the Laplace operator on compact manifolds. This will lead us to define the zeta function that describes the physical models we want to study and to state our main result in Proposition 3 and its corollary. In Section 3, we prove asymptotic expansions at high and low temperature. In Section 4, we study the zeta function introduced in Section 2 for some particular geometries at finite temperature, and we provide formulas for the main thermodynamic functions.

2. The zeta regularized partition function

We recall some known facts about the geometry of the Laplace operator on compact manifolds. Let $(M, \partial M, g)$ be a compact connected Riemannian manifold of dimension D with (possibly empty) boundary ∂M. Let (Δ_M, BC) be the Laplace operator built using the metric g and acting on the opportune space of functions with boundary conditions BC on the boundary of M. If the boundary of M is not empty, Dirichlet or Neumann boundary conditions can be chosen. Let $\text{Sp}(-\Delta_M, BC)$ denotes the spectrum of the associated boundary value problem. Then, $\text{Sp}(-\Delta_M, BC)$ is a discrete set of non negative (real) numbers: $\{\lambda_k\}_{k \in K}$, $K \subseteq \mathbb{N}$, and $\ker(-\Delta_M)$ is a finite set of rank $\mathcal{K}$. Let Sp_0 denotes the set of the positive eigenvalues and K_0 the set of their indices. If ∂M is empty, then $\mathcal{K} = \#ker(-\Delta_M) = 1$, the constant function. In this situation, we have the Weyl formula, that gives the behavior of the large eigenvalues, $\lambda_k \sim \frac{4\pi^2}{(\text{vol}M \text{vol}B_D)^{\frac{2}{D}}} k^{\frac{2}{D}}$, where B_D is the unit ball in $\mathbb{R}^D$, and $\text{vol}B_D = \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)}$, [19] [21] [25] [17], and the heat kernel expansion $\sum_{k \in K} e^{-\lambda_k t} = t^{-\frac{D}{2}} \sum_{j=0}^{\infty} e_j t^{\frac{j}{2}}$, when $t \to 0^+$ [15]. The formulas above hold also if a regular potential term is added to the Laplacian and $e_0 = \frac{\text{vol}M}{(4\pi)^{\frac{D}{2}}}$ [15]. Before introducing the zeta function, we need the following technical lemma, that can be easily deduce from the Young's inequality.

Lemma 1 *For all real positive a and b, integer positive n and k, $n^a + k^b > (nk)^{\frac{ab}{a+b}}$.*

We are now able to introduce the associated zeta function. We want to do this in slightly more general setting, namely we assume a possible mass term, and we consider the operator $-\Delta_M + q$, with real $q \geq 0$. The kernel of this operator depends on q, but its rank is discontinuous at $q = 0$. We will see in the following, that all the results we are able to prove, are true uniformly in q, for all q in some fixed closed interval of the positive real axis. This suggests to deal independently with the zero mass case. On the other side, to simplify notation and avoid to give always two formulas, it is preferable to choose the following alternative approach. Let's introduce the function $\mathcal{K}_q = \begin{cases} 0 & \text{if } q \neq 0 \\ \mathcal{K} & \text{if } q = 0, \end{cases}$, and assume q to be fixed. Then, we can write a unique formula for both the cases, $q = 0$ and $q \neq 0$, that is clearly non smooth in q. Also, we state now once for ever that all the following formulas are smooth and uniform in q for q in any fixed closed interval of the positive real axis.

We first introduce the zeta function at null temperature,

$$\zeta(s; 0, q) = \zeta(s; -\Delta_M + q, BC) = \sum_{k \in K}{}' (\lambda_k + q)^{-s},$$

for $\text{Re}(s) > \frac{D}{2}$, where the notation means that the (possible) zero terms must be omitted in the sum; we will call this function the geometric zeta function associated to the model. Next, consider the product manifold $S^1 \times M$, with the product metric $d^2 t \oplus g$, Laplacian $\Delta = \frac{\partial^2}{\partial^2 t} + \Delta_M$, and periodic boundary conditions $u(0) = u(\beta)$ on S^1 parameterized

by $t \in [0, \beta]$. Then, $\mathrm{Sp}(-\Delta, BC) = \{(2\pi Tn)^2 + \lambda_k\}_{n \in \mathbb{Z}, k \in K}$, and the associated zeta function is§

$$\zeta(s; 2\pi T, q) = {\sum_{(n,k) \in \mathbb{Z} \times K}}' [(2\pi Tn)^2 + \lambda_k + q]^{-s},$$

for $\mathrm{Re}(s) > \frac{D+1}{2}$ by Lemma 1. The zeta regularized partition functions for a (possibly) massive scalar field whose underlying geometry is $(M, \partial M, g)$ is

$$\log Z(T, q) = -\frac{1}{2} \log \det(\rho(-\Delta + q), BC) = \frac{1}{2} \zeta'(0; 2\pi T, q) - \frac{1}{2} \log \rho \zeta(0; 2\pi T, q).$$

As stated in the introduction, our approach is to study the zeta functions as mathematical objects, providing all the zeta invariants, and hence to write down formulas for the partition function of the physical models and to calculate the thermodynamic functions. In the most general setting of this section, all basic information about the zeta function can be obtained using classical methods (see [15] or [26]). We summarize them in Propositions 1 and 2. We also provide the fundamental analytic representation for the zeta function, Proposition 3, that will be the starting point for all the proofs in the following. From now on, we will use the variable $y = 2\pi T$ to simplify notation.

Proposition 1 *The function $\zeta(s; 0, q)$ has an analytic continuation to the whole complex s-plane up to a set of simple poles at the value of $s = \frac{D-l}{2}$, for $l = 0, 1, 2, \ldots$, that are not non-positive integers, with residua:* $\mathrm{Res}_1\left(\zeta(s; 0, q), s = \frac{D-l}{2}\right) = \frac{1}{\Gamma\left(\frac{D-l}{2}\right)} \sum_{j,k \geq 0, j+2k=l} \frac{(-1)^k}{k!} e_j q^k$; *at zero and negative integers, $-m = 0, -1, -2, -3, \ldots$:* $\zeta(-m; 0, q) = (-1)^m m! \sum_{j,k \geq 0, j+2k=2m+D} \frac{(-1)^k}{k!} e_j q^k - \frac{\mathcal{K}_q}{\Gamma(1-m)}$.

Proposition 2 *The function $\zeta(s; y, q)$ has an analytic continuation to the whole complex s-plane, uniformly in y for y in any closed interval of the positive real axis, up to a set of simple poles at the value of $s = \frac{D+1-l}{2}$, for $l = 0, 1, 2, \ldots$, that are not non-positive integers, with residua‖:* $\mathrm{Res}_1\left(\zeta(s; y, q), s = \frac{D+1-l}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{D+1-l}{2}\right)} \frac{1}{y} \sum_{j,k \geq 0, j+2k=l} \frac{(-1)^k}{k!} e_j q^k$; *$s = 0$ is a regular point and $\zeta(0; y, q) = \frac{\sqrt{\pi}}{y} \sum_{j,k \geq 0, j+2k=D+1} \frac{(-1)^k}{k!} e_j q^k - \mathcal{K}_q$.*

Notice in particular the homogeneous case $\zeta(0; 0, 0) = e_D - \mathcal{K}$, $\zeta(0; y, 0) = \frac{\sqrt{\pi}}{y} e_{D+1} - \mathcal{K}$.

Proposition 3 *For all $y > 0$, fixed $q \geq 0$ and uniformly in s near $s = 0$,*

$$\zeta(s; y, q) = \frac{\sqrt{\pi}}{\Gamma(s)} y^{-1} \Gamma\left(s - \frac{1}{2}\right) \zeta\left(s - \frac{1}{2}; 0, q\right) + 2\mathcal{K}_q y^{-2s} \zeta_R(2s) +$$

$$+ \frac{4\pi^s}{\Gamma(s)} y^{-s-\frac{1}{2}} \sum_{n=1}^{\infty} {\sum_{k \in K}}' \left(\frac{n}{\sqrt{\lambda_k + q}}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi n}{y} \sqrt{\lambda_k + q}\right).$$

§ Notice that the same consideration introduced previously about the variable q apply for the variable T, but recalling the discussion outlined in the introduction, we will assume positive definite temperature and will work out the $T = 0$ limit once the right analytic continuation has been achieved.

‖ A further term $\frac{2\mathcal{K}_q}{y}$ appears at $s = \frac{1}{2}$.

<u>Proof</u> Let $a_k = \lambda_k + q$. First, we isolate the (possible) vanishing terms:

$$\zeta(s; y, q) = \sum_{n=-\infty}^{+\infty} \sum_{k \in K}' [(yn)^2 + a_k]^{-s} + 2\mathcal{K}y^{-2s}\zeta_R(2s);$$

the second term needs no further comments, for what concerns the first, we proceed as follows. To start, assume $y > 0$ to be fixed. In such a case, we provide a formula that can be analytically continued in s near $s = 0$, and we prove that such extension is uniform in y for all $y \geq 0$. We apply first the Mellin transform,

$$\sum_{n=-\infty}^{+\infty} \sum_{k \in K}' [(yn)^2 + a_k]^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=-\infty}^{+\infty} e^{-y^2 n^2 t} \sum_{k \in K}' e^{-a_k t} dt,$$

and hence the multi dimensional Poisson summation formula, we get

$$\frac{\sqrt{\pi}}{y\Gamma(s)} \int_0^\infty t^{s-\frac{1}{2}-1} \sum_{k \in K}' e^{-a_k t} dt + \frac{2\sqrt{\pi}}{y\Gamma(s)} \int_0^\infty t^{s-\frac{1}{2}-1} \sum_{n=1}^\infty e^{-\frac{\pi^2 n^2}{y^2 t}} \sum_{k \in K}' e^{-a_k t} dt =$$

the integrals are known and give (where $K_\nu(z)$ is the Bessel function)

$$= \frac{\sqrt{\pi}}{y\Gamma(s)} \Gamma\left(s - \frac{1}{2}\right) \zeta\left(s - \frac{1}{2}; 0, q\right) + \frac{4\pi^s y^{-2-\frac{1}{2}}}{\Gamma(s)} \sum_{n=1}^\infty \sum_{k \in K}' \left(\frac{n}{\sqrt{a_k}}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi n y^{-1}\sqrt{a_k}).$$

The second term is a integral function of s, and it is also easy to see (using classical estimates for the Bessel functions) that convergence of the series is uniform in y for bounded $y \geq 0$. This means that the second term can be analytically extended for all s smoothly in $y \geq 0$. The first term can have poles, but dependence on y and s are clearly distinct. Using the results in Proposition 2, and the known expansion for the Gamma function, we find out that there exists a closed neighborhood of $s = 0$ where there are no poles independently from y, for all $y \geq 0$. This gives the thesis. $\square$

In the following result is the main decomposition of the partition function of the physical model at any given temperature in term of the geometric zeta function.

Corollary 1

$$\log Z(T, q) = \#\ker(-\Delta_M + q) \log T - \log \prod_{k \in K}' \left(1 - e^{-\frac{2\pi}{y}\sqrt{\lambda_k + q}}\right) +$$

$$-\frac{1}{2T}\mathrm{Res}_0\left(\zeta(s; 0, q), s = -\frac{1}{2}\right) - \frac{1}{T}(1 - \log 2)\mathrm{Res}_1\left(\zeta(s; 0, q), s = -\frac{1}{2}\right) +$$

$$-\frac{1}{2}\log \rho\zeta(0; 2\pi T, q),$$

smoothly in $T > 0$ and for all fixed $q \geq 0$.

<u>Proof</u> From Proposition 3, we just have to work out the first term. Recall that $\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3)$, and $\Gamma\left(s - \frac{1}{2}\right) = -2\sqrt{\pi} - 2\sqrt{\pi}[2(1 - \log 2) - \gamma]s + O(s^2)$, and use Proposition 1 to write

$$\zeta\left(s - \frac{1}{2}; 0, q\right) = \frac{c_{-1}}{s} + c_0 + c_1 s + O(s^2) = \frac{R_1}{s} + R_0 + c_1 s + O(s^2),$$

when $s \to 0$, where $R_i = \mathrm{Res}_i\left(\zeta(s; 0, q), s = -\frac{1}{2}\right)$. Then, $\frac{1}{\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right)\zeta\left(s - \frac{1}{2}; 0, q\right) = -2\sqrt{\pi}R_1 - 2\sqrt{\pi}[R_0 + 2(1 - \log 2)R_1]s + O(s^2)$, and the corollary follows. $\square$

3. Low and high temperature expansions

We state our results in the most general setting, using the zeta function introduced in the previous section. The expansions for the partition function at low and high temperature can be immediately obtained as corollaries (For an overview on the physical literature on the thermodynamic on the zero mode and the high and low temperature expansions the interested reader can have a look at [14] and to the references there).

Proposition 4 *For $y \to 0^+$, with fixed $q \geq 0$ and uniformly in s near $s = 0$,*

$$\zeta(s; y, q) = \frac{\sqrt{\pi} y^{-1}}{\Gamma(s)} \Gamma\left(s - \frac{1}{2}\right) \zeta\left(s - \frac{1}{2}; 0, q\right) + 2\mathcal{K}_q y^{-2s} \zeta_R(2s) + O\left(e^{-\frac{1}{y}}\right).$$

<u>Proof</u> This follows immediately from Proposition 3. $\square$

Corollary 2 *For small T, and all fixed $q \geq 0$,*

$$\log Z(T, q) = \#\ker(-\Delta_M + q) \log T - \frac{1}{2T} \mathrm{Res}_0 \left(\zeta(s; -\Delta_M, BC), -\frac{1}{2}\right) +$$

$$-(1 - \log 2)\mathrm{Res}_1 \left(\zeta(s; -\Delta_M, BC), -\frac{1}{2}\right) \frac{1}{T} +$$

$$-\frac{1}{2} \log \rho \left(\frac{1}{\sqrt{\pi} T} \sum_{j,k \geq 0, j+2k=D+1} \frac{(-1)^k}{k!} e_j q^k - \mathcal{K}_q\right) + O\left(e^{-\frac{1}{T}}\right).$$

Proposition 5 *For $y \to +\infty$, with fixed $q \geq 0$, and uniformly in s near $s = 0$,*

$$\zeta(s; y, q) = \zeta(s; 0, q) + \frac{2e_0 \pi^{-\frac{1}{2}+2s-D}}{\Gamma(s)} \Gamma\left(\frac{D+1}{2} - s\right) \zeta_R(D+1-2s) y^{D-2s} + o(y^{D-2s}).$$

<u>Proof</u> Let $a_k = \lambda_k + q$. We first isolate the (possible) vanishing terms as follows:

$$\zeta(s; y, q) = 2 \sum_{n=1}^{\infty} \sum_{k \in K} [(yn)^2 + a_k]^{-s} + \zeta(s; 0, q);$$

let $\sigma = \frac{1}{y^2}$, then: $\zeta^{(1)}(s; y, q) = y^{-2s} 2 \sum_{n=1}^{\infty} \sum_{k \in K} [n^2 + \sigma a_k]^{-s} = y^{-2s} \zeta_\sigma(s; q)$, and we study $\zeta_\sigma(s; q)$. We start assuming $\mathrm{Re}(s) \geq s_1 > \frac{D+1}{2}$, $\sigma \in [\sigma_1, 1]$. Since convergence is absolute we can exchange summation indices as desired. Applying the Mellin transform

$$\zeta_\sigma^{(1)}(s; q) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} 2 \sum_{n=1}^{\infty} e^{-n^2 t} \sum_{k \in K} e^{-\sigma t a_k} dt;$$

splitting the integral at $t = \frac{1}{\sigma}$, in the first integral, $\sigma t \leq 1$, and hence we can use the heat kernel expansion for $-\Delta_M$, while the second integral gives a regular function of s for all s. For the first term, we get (recall $\alpha = \frac{D}{2}$)

$$\frac{\sigma^{-\alpha}}{\Gamma(s)} \sum_{j=0}^{\infty} e_j \sigma^{\frac{j}{2}} 2 \sum_{n=1}^{\infty} \int_0^{\frac{1}{\sigma}} t^{s+\frac{j}{2}-\alpha-1} e^{-n^2 t} dt;$$

notice that we can not use the Poisson formula here, since we are interested in the small σ expansion. Now, for each j,

$$\sum_{n=1}^{\infty} \int_0^{\frac{1}{\sigma}} t^{s+\frac{j}{2}-\alpha-1} e^{-n^2 t} dt = \Gamma\left(s - \alpha + \frac{j}{2}\right) \zeta_R(2s - 2\alpha + j) - \sum_{n=1}^{\infty} \int_{\frac{1}{\sigma}}^{\infty} t^{s+\frac{j}{2}-\alpha-1} e^{-n^2 t} dt.$$

The first term can just have simple poles at $\alpha - \frac{j}{2}$ and $\alpha - \frac{1-j}{2}$. In fact the Gamma function can have a pole only at non positive integer argument, but for even negative integers the zeta function vanishes. Thus, for each j and all $s \neq \alpha - \frac{j}{2}, \alpha - \frac{1-j}{2}$,

$$\lim_{\sigma \to 0^+} 2 \sum_{n=1}^{\infty} \int_0^{\frac{1}{\sigma}} t^{s+\frac{j}{2}-\alpha-1} e^{-n^2 t} dt = 2\Gamma\left(s - \alpha + \frac{j}{2}\right) \zeta_R(2s - 2\alpha + j),$$

where the analytic extension of the above function of s is intended for $\operatorname{Re}(s) < \alpha + \frac{j}{2}$. For small σ, we can write

$$\frac{1}{\Gamma(s)} \int_0^{\frac{1}{\sigma}} t^{s-1} \sum_{n=1}^{\infty} e^{-n^2 t} \sum_{k=0}^{\infty} e^{-\sigma t a_k(x)} dt = \sum_{j=0}^{\infty} e_j \sigma^{\frac{j}{2}-\alpha} \frac{\Gamma\left(s - \alpha + \frac{j}{2}\right) \zeta_R(2s - 2\alpha + j)}{\Gamma(s)} + o(1)$$

and this is a regular function of s for s closed to 0, independently from σ, and hence gives the desired result. The second term with $x = \sigma t$, and using the Poisson formula becomes

$$\sigma^{-s} \int_1^{\infty} x^{s-1} 2 \sum_{n=1}^{\infty} e^{-\frac{n^2 x}{\sigma}} \sum_{k=0}^{\infty} e^{-x a_k(x)} dx = \sigma^{-s} O(e^{-\frac{1}{\sigma}}),$$

for all s, since the second factor is a regular function of s for all s. Collecting and recasting the correct functions we get the thesis, where we put in evidence the leading term and use the Riemann's functional equation to rewrite its coefficient. $\square$

Corollary 3 *For large T,*

$$\log Z(T, q) = \frac{\operatorname{vol} M}{\pi^{\frac{D+1}{2}}} \Gamma\left(\frac{D+1}{2}\right) \zeta_R(D+1) T^D + o(T^D).$$

4. Finite temperature results for some particular geometries

We turn now our attention to the case of a scalar field of rest mass q in two fixed geometries: a cubic box $\mathcal{B}_D$ of edge l and a D-dimensional torus $\mathcal{T}_D = S^1_{l/\pi} \times \ldots \times S^1_{l/\pi}$, at any fixed value of the temperature T. The operator A in the action is the negative of the D-dimensional Laplacian Δ plus a constant potential in the Euclidean space and the difference in the two models is in the boundary conditions: Dirichlet boundary conditions on the boundary of the cube and periodic boundary condition for the closed domain. A complete system of eigenvalues is then: $\lambda_{n,k}(T, q, l) = (2\pi T n)^2 + \frac{\pi^2}{l^2}|k|^2 + q$, where $n \in \mathbb{Z}$ is an integer and $k = (k_1, \ldots, k_D)$ is a positive integral vector in $(\mathbb{N}_0)^D$ in the first case, but is any integral vector in $\mathbb{Z}^D$ in the second one ($2l$ is the length of the circles in the torus in the second case). The zeta functions are

$$\zeta_{\mathcal{B}_D}(s; 2\pi T, q, l) = \sum_{(n,k) \in \mathbb{Z} \times (\mathbb{N}_0)^D} \left[(2\pi T n)^2 + \frac{\pi^2}{l^2}|k|^2 + q\right]^{-s},$$

$$\zeta_{\mathcal{T}_D}(s; 2\pi T, q, l) = \sideset{}{'}\sum_{(n,k)\in\mathbb{Z}^{D+1}} \left[(2\pi T n)^2 + \frac{\pi^2}{l^2}|k|^2 + q \right]^{-s},$$

for $\mathrm{Re}(s) > \frac{D+1}{2}$. Using the results of Section 2, we just have to study the associated geometric zeta functions, namely

$$\zeta_{\mathcal{B}_D}(s; 0, q, l) = \sum_{k\in(\mathbb{N}_0)^D} \left[\frac{\pi^2}{l^2}|k|^2 + q \right]^{-s}, \quad \zeta_{\mathcal{T}_D}(s; 0, q, l) = \sideset{}{'}\sum_{k\in\mathbb{Z}^D} \left[\frac{\pi^2}{l^2}|k|^2 + q \right]^{-s},$$

for $\mathrm{Re}(s) > \frac{D}{2}$. This is the aim of this section, that is subdivided in three parts. In the first we introduce a multidimensional generalization of the Riemann zeta function useful to describe the geometric zeta functions, subsequently we calculate the main zeta invariants, and in the last part we give the partition function and the thermodynamic functions. From now on we will assume $D \geq 1$ if not otherwise stated.

4.1. Multidimensional quadratic zeta functions

Let q^2 be a complex constant that is not real and negative, n a D-dimensional vector with integer components in $\mathbb{Z}^D$, and A a real symmetric matrix of rank D with positive definite associated quadratic form, then we can introduce the functions:

$$\xi_D(s; q) = \sum_{n\in(\mathbb{N}_0)^D} (|n|^2 + q^2)^{-s},$$

$$\zeta_D(s; A, q) = \sum_{n\in\mathbb{Z}^D} (n^T A n + q^2)^{-s} = q^{-2s} + \hat{\zeta}_D(s; n^T A n, q)$$

when $\mathrm{Re}(s) > \frac{D}{2}$. Notice that the definition introduced for $\zeta_D(s; A, q)$ is ad hoc to avoid problems for the homogeneous case $q = 0$. In fact, while the definition given for $\xi_D(s; q)$ extends to $q = 0$, in the Eisenstein sum one must omit the null vector when $q = 0$. With the definition above, this simply means to omit the q^{-2s} term in the homogeneous case. We will use the notation $\zeta_D(s; q)$ for $\zeta_D(s; I, q)$ in the following. Notice in particular that $\zeta_0(s; q) = q^{-2s}$ and $2\xi_1(s; 0) = \hat{\zeta}_1(s; 0) = 2\zeta_R(2s)$. Actually, the results we are going to give for the Epstein zeta functions hold true for a large class of zeta functions, that we introduce now. Let $n \in \mathbb{Z}^D$ and A be as above, b and $x = (x_i)$ be real D dimensional vectors, and assume $0 \leq x_i < 1$. Then, we define the functions:

$$\zeta_D(s; A, b, x, q) = \sum_{n\in\mathbb{Z}^D} [(n+x)^T A(n+x) + b^T n + q^2]^{-s},$$

$$\hat{\zeta}_D(s; A, b, x, q) = \sum_{n\in\mathbb{Z}_0^D} [(n+x)^T A(n+x) + b^T n + q^2]^{-s},$$

when $\mathrm{Re}(s) > \frac{D}{2}$. Notice that q^2 must be non vanishing in the definition of $\zeta_D(s; A, b, x, q)$. A lot is known about these multidimensional zeta functions, in particular the homogeneous case, namely the Epstein zeta function, has been deeply investigated (see [31] for a good overview, or locally cited references). Here, we collect a series of results that seem more interesting and useful for the present purposes. As we

will see, it is easier to get more general results for the Epstein series, and this is essentially due to the possibility of more effective use of the Poisson summation formula. All the proofs are based on classical techniques, namely the Mellin transform and the Poisson summation formula; since these tools were used for all the proofs in the previous sections we omit to give details here and refer the interested reader to the literature available for a deeper account on this subject.

We begin by introducing some analytic representations. These will be useful to get all information about the analytic extensions of the zeta functions, as well as when calculations are involved to evaluate them at some particular value.

Lemma 2 $\quad \xi_D(s;q) = -\frac{1}{2}\xi_{D-1}(s;q) + \frac{\sqrt{\pi}}{\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right)\xi_{D-1}\left(s - \frac{1}{2};q\right) +$

$$+\frac{4\pi^s}{\Gamma(s)}\sum_{(n,k)\in(\mathbb{N}_0)^D}\left(\frac{\sqrt{|k|^2+q^2}}{n}\right)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi n\sqrt{|k|^2+q^2}).$$

For the Epstein type functions, we have the following lemma when $q \neq 0$:

Lemma 3

$$\zeta_D(s;A,0,x,q) = \frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}}\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)}q^{-2s+D} + \frac{2\pi^s}{\sqrt{\det A}\Gamma(s)}q^{\frac{D}{2}-s}\sum_{n\in\mathbb{Z}_0^D}\left(\frac{\sqrt{n^TA^{-1}n}}{q}\right)^{s-\frac{D}{2}}K_{s-\frac{D}{2}}(2\pi q\sqrt{n^TA^{-1}n}),$$

$$\zeta_D(s;A,b,0,q) = \frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}}\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)}\left(q^2 - \frac{1}{4}b^TA^{-1}B\right)^{\frac{D}{2}-s} +$$

$$+\frac{2\pi^s}{\sqrt{\det A}\Gamma(s)}\sum_{n\in\mathbb{Z}_0^D}\left(\frac{n^TA^{-1}n}{q^2-\frac{1}{4}b^TA^{-1}b}\right)^{\frac{1}{2}\left(s-\frac{D}{2}\right)}K_{s-\frac{D}{2}}(\pi\sqrt{4q^2-b^TA^{-1}b}\sqrt{n^TA^{-1}n}), \text{ if } 4q^2 - b^TA^{-1}b > 0.$$

When $q = 0$, we need some more notation. Let $a_{i,j}$ be the elements of A. Let $A_{1,1}$ the minor of $a_{1,1}$ in A, and A_1 denotes the $D-1$-column vector whose elements are the elements of the first line of $A_{1,1}$. Let $\hat{b}_1$ and $\hat{x}_1$ be the $D-1$ vectors whose elements are the last $D-1$ elements of b and x, respectively. Let B be the $D-1$ square matrix whose elements are $b_{i-1,j-1} = a_{i,j} - \frac{a_{1,i}a_{1,j}}{a_{1,1}}$, where the indices i and j run from 2 to $D-1$.

Lemma 4 $\quad \hat{\zeta}_D(s;A,0,x,0) = \hat{\zeta}_1(s;a_{1,1},0,x_1,0) + \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{a_{1,1}}\Gamma(s)}\hat{\zeta}_{D-1}\left(s-\frac{1}{2};B,0,\hat{x}_1,0\right) + \frac{4\pi^s}{\Gamma(s)}a_{1,1}^{-\frac{1}{2}\left(s-\frac{1}{2}\right)}\times$

$$\times\sum_{n_1=1}^{\infty}\sum_{n\in\mathbb{Z}_0^{D-1}}\cos 2\pi\left[n_1x_1 + \frac{1}{a_{1,1}}A_1(n+\hat{x}_1)n_1\right]\frac{n_1^{s-\frac{1}{2}}}{(n^TBn)^{\frac{1}{2}\left(s-\frac{1}{2}\right)}}K_{s-\frac{1}{2}}\left(\frac{2\pi n_1}{\sqrt{a_{1,1}}}\sqrt{n^TBn}\right),$$

$$\hat{\zeta}_D(s;A,b,0,0) = \hat{\zeta}_1(s;a_{1,1},b_1,0,0) + \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{a_{1,1}}\Gamma(s)}\hat{\zeta}_{D-1}\left(s-\frac{1}{2};B,0,\frac{b_1}{a_{1,1}}A_1,-\frac{b_1^2}{4a_{1,1}}\right) + \frac{4\pi^s}{\Gamma(s)}a_{1,1}^{-\frac{1}{2}\left(s-\frac{1}{2}\right)}\times$$

$$\times\sum_{n_1=1}^{\infty}\sum_{n\in\mathbb{Z}_0^{D-1}}\cos\left[\frac{\pi}{2a_{1,1}}(2A_1n+b_1)n_1\right]\frac{n_1^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(\frac{2\pi n_1}{\sqrt{a_{1,1}}}\sqrt{n^TA_{1,1}n-\frac{1}{4a_{1,1}}(2A_1n+b_1)^2}\right)}{\left(n^TA_{1,1}n-\frac{1}{4a_{1,1}}(2A_1n+b_1)^2\right)^{\frac{1}{2}\left(s-\frac{1}{2}\right)}} =$$

$$\hat{\zeta}_{D-1}(s;A_{1,1},\hat{b}_1,0,0) + \frac{\pi^{\frac{D-1}{2}}\Gamma\left(s-\frac{D-1}{2}\right)}{\sqrt{\det A_{1,1}}\Gamma(s)}\hat{\zeta}_1\left(s-\frac{D-1}{2};\frac{\det A}{\det A_{1,1}},b_1 - A_1^TA_{1,1}^{-1}\hat{b}_1,0,-\frac{1}{4}\hat{b}_1^TA_{1,1}^{-1}\hat{b}_1\right) + \frac{2\pi^2}{\sqrt{\det A_{1,1}}}\times$$

$$\sum_{n\in\mathbb{Z}_0^{D-1}}e^{\pi i n^TA_{1,1}^{-1}(2n_1A_1+\hat{b}_1)}(n^TA_{1,1}^{-1}n)^{\frac{1}{2}\left(s-\frac{D-1}{2}\right)}\frac{K_{s-\frac{D-1}{2}}\left(2\pi\sqrt{n^TA_{1,1}^{-1}n}\sqrt{\frac{\det A}{\det A_{1,1}}n_1^2+(b_1-A_1^TA_{1,1}^{-1}\hat{b}_1)n_1-\frac{1}{4}\hat{b}_1^TA_{1,1}^{-1}\hat{b}_1}\right)}{\left(\frac{\det A}{\det A_{1,1}}n_1^2+(b_1-A_1^TA_{1,1}^{-1}\hat{b}_1)n_1-\frac{1}{4}\hat{b}_1^TA_{1,1}^{-1}\hat{b}_1\right)^{\frac{1}{2}\left(s-\frac{D-1}{2}\right)}},$$

where the last two representations hold if $-\frac{b_1^2}{4a_{1,1}}$ *or* $-\frac{1}{4}\hat{b}_1^TA_{1,1}^{-1}\hat{b}_1$ *are not negative integers.*

More Chowla-Selberg type formulas [6] [5] can be found in [31]. We also recall the important reflection formula [27] [30]

Lemma 5 $\qquad \pi^{-s}\Gamma(s)\hat{\zeta}_D(s; A, 0, 0, 0) = \frac{\pi^{s-\frac{D}{2}}}{\sqrt{\det A}}\Gamma\left(\frac{D}{2} - s\right)\hat{\zeta}_D\left(\frac{D}{2} - s; A^{-1}, 0, 0, 0\right).$

Corollary 4 $\qquad \zeta'_D(0; A, 0, 0, q) = \begin{cases} \frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}}\Gamma\left(-\frac{D}{2}\right)q^D & D \text{ odd} \\ \frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}}\frac{(-1)^{\frac{D}{2}}}{\frac{D}{2}!}q^D & D \text{ even} \end{cases} +$

$$+ \frac{2\pi^{\frac{D}{2}}}{\sqrt{\det A}}\sum_{n \in \mathbb{Z}_0^D}(n^T A^{-1}n)^{-\frac{D}{4}}K_{-\frac{D}{2}}(2\pi q\sqrt{n^T A^{-1}n}).$$

See also [6] and [31] for the Kronecker limit formula.

Corollary 5 *For* $n = 1, 2, 3, \ldots$: $\hat{\zeta}_D(-n; A, 0, 0, 0) = 0$; *for* $n = 0, 1, 2, \ldots$, $q \neq 0$

$$\zeta_D(-n; A, 0, 0, q) = q^{2n} + \hat{\zeta}_D(-n; A, 0, 0, q) = \begin{cases} 0 & D \text{ odd} \\ \frac{(-1)^{-n}\frac{D}{2}!}{(n + \frac{D}{2})!}\frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}}q^{2n+D} & D \text{ even}; \end{cases}$$

Eventually, using the representation introduced in the previous lemmas, or using classical methods, we get all information about poles, residua and particular values.

Lemma 6 *The function* $\xi_D(s; q)$ *extends analytically to the whole complex plane up to simple poles at* $s = \frac{D}{2}, \frac{D-1}{2}, \ldots, \frac{1}{2}$ *and* $s = -\frac{1}{2} - j$, $j = 0, 1, 2, \ldots$, *with residua:*

$\text{Res}_1\left(\xi_D(s; q), -\frac{k}{2}\right) = \frac{(-1)^D}{2^D\Gamma\left(-\frac{k}{2}\right)}\sum_{j=1}^{D}\sum_{i=0, 2i=j+k}^{\infty}\frac{(-1)^{i+j}}{i!}\begin{pmatrix} D \\ j \end{pmatrix}\pi^{\frac{j}{2}}q^{2i}$, $k = -D, -D +$

$1, \ldots, k \notin 2\mathbb{N}$; $\xi_D(0; q) = \frac{(-1)^D}{2^D} + \frac{(-1)^D}{2^D}\sum_{j=1}^{\frac{D}{2}}\begin{pmatrix} D \\ 2j \end{pmatrix}\frac{(-1)^j\pi^j q^{2j}}{\Gamma(j+1)}.$

Notice that in the homogeneous case there are poles at $s = \frac{1}{2}, \ldots, \frac{D}{2}$, with residuum

$\text{Res}_1\left(\xi_D(s; 0), s = \frac{i}{2}\right) = \frac{(-1)^{D+i}}{2^D\Gamma\left(\frac{i}{2}\right)}\begin{pmatrix} D \\ i \end{pmatrix}\pi^{\frac{i}{2}}$, $i = 1, 2, \ldots, D$, $\xi_D(0; 0) = \frac{(-1)^D}{2^D}.$

Lemma 7 *The analytic continuation of* $\hat{\zeta}_D(s; A, 0, 0, q)$ *and* $\zeta_D(s; A, 0, 0, q)$ *are regular on the whole complex s-plane up to a simple poles at* $s = \frac{D}{2} - j$, $j = 0, 1, 2, \ldots$ *if* D *is odd and* $s = \frac{D}{2} - j$, $j = 0, 1, 2, \ldots, \frac{D}{2} - 1$ *if* D *is even, respectively. The residua are, for both the functions,* $\text{Res}_1\left(\hat{\zeta}_D(s; A, 0, 0, q), s = \frac{D}{2} - j\right) = \frac{(-1)^j}{j!}\frac{\pi^{\frac{D}{2}}q^{2j}}{\sqrt{\det A}\Gamma\left(\frac{D}{2} - j\right)}$, j *as before.*

Moreover: $\zeta_D(0; A, 0, 0, q) = 1 + \hat{\zeta}(0; A, 0, 0, q) = \begin{cases} 0 & D \text{ odd} \\ \frac{(-\pi)^{\frac{D}{2}}q^D}{\frac{D}{2}!} & D \text{ even} \end{cases}.$

Notice that in the homogeneous case the unique pole is at $s = \frac{D}{2}$ with residuum $\frac{\pi^{\frac{D}{2}}}{\sqrt{\det A}\Gamma\left(\frac{D}{2}\right)}$, and $\hat{\zeta}(0; A, 0, 0, 0) = -1$ for all D.

4.2. Dirichlet boundary conditions

In this case, the spectrum of $-\Delta_M$ is positive definite, and we can write the geometric zeta function using the function $\xi_D(s;q)$ just introduced: $\zeta_{\mathcal{B}_D}(s;0,q,l) = \frac{l^{2s}}{\pi^{2s}}\xi_D\left(s;\frac{\sqrt{q}l}{\pi}\right)$. Notice that this extends continuously to the homogeneous case $q = 0$. Using Propositions 2 and 3 in Section 2, and Lemma 6 of the previous part, we get:

Proposition 6 *The function $\zeta_{\mathcal{B}_D}(s;y,q,l)$ extends analytically to a regular function on the whole complex s-plane up to simple poles for all the $s = \frac{D+1-j}{2}$, with $j = 0,1,2,\ldots$, that are not non positive integers, with residua ($k = -D-1, -D, -D+1, \ldots, k \notin \mathbb{N}$):*

$$\mathrm{Res}_1\left(\zeta_{\mathcal{B}_D}(s;y,q,l), s = -\frac{k}{2}\right) = \frac{(-1)^D \pi^{k+1}}{2^D l^{k+1}\Gamma\left(-\frac{k}{2}\right)y}\sum_{i=0}^{D}\sum_{j=0,2j-i-1=k}^{\infty}\binom{D}{i}\frac{(-1)^{i+j}l^{2j}q^j}{j!\pi^{2j-\frac{i+1}{2}}}.$$

Notice that in the homogeneous case the poles are at $s = \frac{D+1}{2}, \frac{D}{2}, \ldots \frac{1}{2}$.

Proposition 7 *For all $y > 0$, fixed $q \geq 0$ and $l > 0$, and uniformly in s near $s = 0$,*

$$\zeta_{\mathcal{B}_D}(s;y,q,l) = \frac{l^{2s-1}}{\pi^{2s-1}}\frac{\sqrt{\pi}}{\Gamma(s)}y^{-1}\Gamma\left(s-\frac{1}{2}\right)\xi_D\left(s-\frac{1}{2};\frac{\sqrt{q}l}{\pi}\right) +$$

$$+\frac{4\pi^s}{\Gamma(s)}y^{-s-\frac{1}{2}}\sum_{(n,k)\in(\mathbb{N}_0)^{D+1}}\left(\frac{n}{\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(\frac{2\pi n}{y}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}\right).$$

As a corollary, we give the formula for the derivative at $s = 0$. Some care is necessary to deal with the first term when $q \neq 0$. In this case in fact, the function $\xi_D(s;q)$ has a pole at $s = -\frac{1}{2}$. Proceeding as in the proof of the corollary to Proposition 3, we get

Corollary 6 *For all fixed $q \geq 0$ and $l > 0$,*

$$\zeta_{\mathcal{B}_D}(0;y,q,l) = -\frac{2\pi^2}{l}R_1\frac{1}{y} = \frac{(-1)^D}{2^D y}\sum_{i=0}^{\left[\frac{D-1}{2}\right]}\binom{D}{2i+1}\frac{(-1)^i}{(i+1)!}\frac{l^{2i+1}q^{i+1}}{\pi^i},$$

$$\zeta'_{\mathcal{B}_D}(0;y,q,l) = A_D(q,l)\frac{2\pi}{y} - 2\log\prod_{k\in(\mathbb{N}_0)^D}\left(1 - \mathrm{e}^{-\frac{2\pi}{y}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right),$$

where $A_D(q,l) = -\frac{\pi}{l}[R_0 + (2 - \log 4\pi + \log l)R_1]$, $R_i = \mathrm{Res}_i\left(\xi_D\left(s;\frac{\sqrt{q}l}{\pi}\right), s = -\frac{1}{2}\right)$.

Notice that in the homogeneous case the value at $s = 0$ is 0 for all D. Also, when $q = 0$, $\xi_D(s;0)$ is regular at $s = -\frac{1}{2}$; thus, an explicit formula for the constant when $q = 0$ is $A_D(0,l) = -\frac{\pi}{l}\xi_D\left(-\frac{1}{2};0\right)$, but it is more complicate otherwise. Using Lemma 2, if $q = 0$, when $D = 1$ we get twice the Riemann zeta function at $s = -1$, while for higher D only numerical evaluations are possible and give: $\xi_1\left(-\frac{1}{2};0\right) = \zeta_R(-1) = -\frac{1}{12}$, $\xi_2\left(-\frac{1}{2};0\right) = 0.026127$, $\xi_3\left(-\frac{1}{2};0\right) = -0.010015$. Notice that these results can be easily obtained using the representations introduced in Lemma 2, since the series there converges very fast. The same computation can be made using the reflection formula

for the multidimensional zeta function given in Lemma 5, but in that case the series converges very slowly and a much longer computation is necessary.

We conclude this part with some remarks on the well know cases $D = 0$ and $D = 1$. When $D = 0$ [29],

$$z_1(s; y, q) = \zeta_{\mathcal{B}_0}(s; y, q, \pi) = \sum_{n \in \mathbb{Z}} (y^2 n^2 + q)^{-s} = q^{-s} + 2y^{-2s} \xi_1 \left(s; \frac{\sqrt{q}}{y} \right);$$

for $q \neq 0$, $z_1(s; y, q)$ has a simple pole at $s = \frac{1}{2} - j$, $j = 0, 1, \ldots$, with residuum $\frac{(-1)^j}{j!} \frac{\sqrt{\pi} q^j}{\Gamma(\frac{1}{2} - j) y}$, and $z_1(0; y, q) = 0$, $z_1'(0, y; q) = -2 \log 2 \text{sh} \frac{\pi \sqrt{q}}{y}$. For $q = 0$, $z_1(s; y, 0)) = 2y^{-2s} \zeta_R(2s)$, has a single pole at $s = \frac{1}{2}$ with residuum $\frac{2}{y}$, and $z_1(0; y, 0) = -1$, $z'(0; y, 0) = 2 \log y - 2 \log 2\pi$.

When $D = 1$, we have the zeta function associate to the Laplacian plus a constant potential on a cylinder [32] [29],

$$z_2(s; y, q) = \zeta_{\mathcal{B}_1}(s; y, q, \pi) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} (y^2 n^2 + k^2 + q)^{-s},$$

and $z_2(0; y, q) = -\frac{\pi q}{2y}$, $z_2'(0; y, q) = \frac{\pi}{6} \frac{1}{y} - (\gamma - \log 2) \frac{\pi q}{y} - 2 \log \prod_{n=1}^{\infty} \left(1 - e^{-\frac{2\pi}{y} \sqrt{n^2 + q^2}} \right) - \frac{2\pi}{y} \sum_{j=2}^{\infty} \binom{\frac{1}{2}}{j} \zeta_R(2j - 1) q^j$. In particular, when $q = 0$, the first term of the s-expansion near $s = 0$ of such zeta function is well known

$$z_2'(0; y, 0) = \frac{\pi}{6} \frac{1}{y} - 2 \log \prod_{n=1}^{\infty} \left(1 - e^{-\frac{2\pi}{y} n} \right) = -2 \log \eta \left(\frac{i}{y} \right),$$

where $\eta(z)$ is the Dedekind eta function. In such case, it is very easy to pass from the small y to the large y expansion using the well known modular transformation of the eta function, namely $\eta \left(-\frac{1}{\tau} \right) = \sqrt{\frac{\tau}{i}} \eta(\tau)$. It is also easy to see how the presence of a non homogeneous term breaks this symmetry. A deeper investigation about this point is performed in the next part.

A final observation is about the large y expansion. As just noticed, this can be immediately worked out when $D = 1$. For higher values of D, it is harder. In fact, we can not use the analytic representation given in Proposition 7 for large y neither we know how the zeta function behaves under the modular transformation $y \to \frac{1}{y}$. Despite that, a direct approach is still possible for fixed D, consisting in expressing the zeta function in dimension D recursively in terms of the zeta function in dimension $D - 1$, and using the known behavior in dimension $D = 1$. Calculations are tedious but straightforward; the leading term in the expansion is consistent with the one obtained using Proposition 5, but now we can get further terms.

4.3. Periodic boundary conditions

Recall that in the present case the spectrum of $-\Delta_M$ is not always positive definite; more precisely, $\mathcal{K}_q = 1$ if $q = 0$, but it vanishes if $q > 0$. We need the functions ζ_D and

$\hat{\zeta}_D$ of 4.1 in order to write the geometric zeta function: $\zeta_{T_D}(s;0,q,l) = \frac{l^{2s}}{\pi^{2s}}\zeta_D\left(s;\frac{\sqrt{q}l}{\pi}\right) = q^{-2s} + \frac{l^{2s}}{\pi^{2s}}\hat{\zeta}_D\left(s;\frac{\sqrt{q}l}{\pi}\right)$. Due to the presence of the first term, it is clear how this expression does not extend to the homogeneous case. In the present situation it is easier to deal with the two cases independently. This does not affect the poles, hence we can state the following unique result using Propositions 2 and 3 in Section 2, and Lemma 4 of 4.1.

Proposition 8 *The functions $\zeta_{T_D}(s;y,q,l)$ and $\hat{\zeta}_{T_D}(s;y,q,l)$ extend analytically to a regular function on the whole complex s-plane up to simple poles for all the $s = \frac{D+1}{2} - j$, with $j = 0,1,2,\ldots$, if D is even, and $j = 0,1,2,\ldots,\frac{D+1}{2}-1$, if D is odd. The residua are for both the functions:* $\operatorname{Res}_1\left(\zeta_{T_D}(s;y,q,l), s = \frac{D+1}{2}-j\right) = \frac{(-q)^j}{j!\Gamma\left(\frac{D+1}{2}-j\right)}\frac{l^D}{\pi^{\frac{D-1}{2}}}\frac{1}{y}$.

Notice that in the homogeneous case, the unique pole is at $s = \frac{D+1}{2}$ with residuum $\frac{l^D}{\pi^{\frac{D-1}{2}}\Gamma\left(\frac{D+1}{2}-j\right)}\frac{1}{y}$. An analytic representation analogous to the one stated in Proposition 7 of the previous part is

Proposition 9 *For all $y > 0$, $l > 0$, and uniformly in s near $s = 0$,*

$$\zeta_{T_D}(s;y,q,l) = \frac{l^{2s-1}}{\pi^{2s-1}}\frac{\sqrt{\pi}}{\Gamma(s)}y^{-1}\Gamma\left(s-\frac{1}{2}\right)\zeta_D\left(s-\frac{1}{2};\frac{\sqrt{q}l}{\pi}\right) +$$

$$+\frac{4\pi^s}{\Gamma(s)}y^{-s-\frac{1}{2}}\sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}^D}\left(\frac{n}{\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(\frac{2\pi n}{y}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}\right),$$

uniformly in q for q in any closed subset of the positive real axis, while

$$\zeta_{T_D}(s;y,0,l) = \frac{l^{2s-1}}{\pi^{2s-1}}\frac{\sqrt{\pi}}{\Gamma(s)}y^{-1}\Gamma\left(s-\frac{1}{2}\right)\hat{\zeta}_D\left(s-\frac{1}{2};0\right) + 2y^{-2s}\zeta_R(2s) +$$

$$+\frac{4\sqrt{\pi}l^{s-\frac{1}{2}}}{\Gamma(s)}y^{-s-\frac{1}{2}}\sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}_0^D}\left(\frac{n}{|k|}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(\frac{2\pi^2 n}{ly}|k|\right).$$

Corollary 7 *Uniformly in $q > q_0 > 0$:*

$$\zeta_{T_D}(0;y,q,l) = -\frac{2\pi^2}{l}R_1\frac{1}{y} = \begin{cases} 0 & D \text{ even} \\ \frac{(-q)^{\frac{D+1}{2}}l^D}{\frac{D+1}{2}!\pi^{\frac{D-1}{2}}}\frac{1}{y} & D \text{ odd,} \end{cases}$$

$$\zeta_{T_D}'(0;y,q,l) = B_D(q,l)\frac{2\pi}{y} - 2\log\prod_{k\in\mathbb{Z}^D}\left(1-e^{-\frac{2\pi}{y}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right);$$

$$\zeta_{T_D}(0;y,0,l) = -\frac{2\pi^2}{l}\hat{R}_1\frac{1}{y} + 2\zeta_R(0) = -1,$$

$$\zeta_{T_D}'(0;y,0,l) = \hat{B}_D(0,l)\frac{2\pi}{y} + 2\log\frac{y}{2\pi} - 2\log\prod_{k\in\mathbb{Z}_0^D}\left(1-e^{-\frac{2\pi^2}{ly}|k|}\right),$$

where $B_D(q,l) = -\frac{\pi}{l}[R_0 + (2 - \log 4\pi + \log l)R_1]$, $R_i = \operatorname{Res}_i\left(\zeta_D\left(s;\frac{\sqrt{q}l}{\pi}\right), s = -\frac{1}{2}\right)$, and similarly for the hatted ones.

We get simple expressions for the constants when D is even or when $q = 0$. In particular, in the second case, $\hat{B}_D(0, l) = -\frac{\pi}{l}\hat{\zeta}_D\left(-\frac{1}{2}; 0\right)$. Using Lemma 4 and numerical evaluations: $\hat{\zeta}_1\left(-\frac{1}{2}; 0\right) = 2\zeta_R(-1) = -\frac{1}{6}$, $\hat{\zeta}_2\left(-\frac{1}{2}; 0\right) = -0.2286$, $\hat{\zeta}_3\left(-\frac{1}{2}; 0\right) = -0.26493$.

Of particular interest is the case $D = 1$. As stated in the previous part, this is related with the Dedekind eta function, $\eta(z)$. Namely, assuming $l = \pi$ for simplicity,

$$\lim_{q\to 0^+}\left[\zeta'_{\mathcal{T}_1}(0; y, q, \pi) + \log q\right] = 2\log y - 2\log 2\pi - 4\log\eta\left(\frac{i}{y}\right).$$

This suggest to define the function

$$\eta(\tau, q) = -e^{\pi i\tau B_2(q)}\left(1 - e^{2\pi i\tau q}\right)\prod_{n=1}^{\infty}\left(1 - e^{2\pi i\tau\sqrt{n^2+q^2}}\right)^2,$$

for real positive q and complex τ with positive imaginary part. It is easy to check that $\lim_{q\to 0^+}\frac{\eta(\tau,q)}{2\pi i\tau q} = \eta^2(\tau)$. It is also easy to realize that the presence of the non homogeneous term q, breaks modularity. On the other side, the modular transformation for the Dedekind eta function can be deduced using the symmetry in the definition of the zeta function $\zeta_{\mathcal{T}_1}(s; y, q^2, \pi)$ under the exchange of the summation indices in the first term of the s-expansion near $s = 0$. Using the same symmetry for the function $\eta(iy, q)$, we get instead of the modular transformation the following relation

$$\log\eta\left(\frac{i}{y}, qy\right) = \log\eta(iy, q) - \pi q^2 y\log y + 2\pi y\sum_{j=2}^{\infty}\binom{\frac{1}{2}}{j}\zeta_R(2j-1)q^{2j} + \frac{2\pi}{y}\sum_{j=2}^{\infty}\binom{\frac{1}{2}}{j}\zeta_R(2j-1)(qy)^{2j}.$$

Since the behavior of $\eta(iy, q)$ for large y is clear, the above expression can be used (exactly as it was for the Dedekind zeta function) to deduce the behavior for small y. We get, for $y \to 0^+$:

$$\eta(iy, q) = -\frac{\pi}{6y} + \pi q^2 y\log y + \pi(q+1) - \left[\sum_{j=2}^{\infty}\binom{\frac{1}{2}}{j}\zeta_R(2j-1)q^{2j} + \pi q^2\right]y - \frac{\pi y^3}{4}\zeta_R(3) + O(y^4).$$

4.4. Thermodynamic functions

We write now explicit formulas for the partition function of the models introduced in 4.1. Such formulas can be used to get explicit expressions for all the thermodynamic functions. In particular, the behavior for low and high temperature are given. The partition function for a massive scalar thermal radiation at temperature T in a box of volume l^D and on the torus $\mathcal{T}_D$ are, for any fixed positive l,

$$\log Z_{\mathcal{B}_D}(T, q, l) = \frac{1}{2}A_D(q, l)\frac{1}{T} - \log\prod_{k\in(\mathbb{N}_0)^D}\left(1 - e^{-\frac{1}{T}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right) +$$

$$+\frac{(-1)^{D+1}}{2^{D+2}}\log\rho\frac{1}{T}\sum_{i=0}^{\left[\frac{D-1}{2}\right]}\binom{D}{2i+1}\frac{(-1)^i}{(i+1)!}\frac{l^{2i+1}q^{i+1}}{\pi^i},$$

$$\log Z_{\mathcal{T}_D}(T, q, l) = \frac{1}{2}B_D(q, l)\frac{1}{T} - \log\prod_{k\in\mathbb{Z}^D}\left(1 - e^{-\frac{1}{T}\sqrt{\frac{\pi^2}{l^2}|k|^2+q}}\right) +$$

$$-\frac{1}{2}\log\rho\frac{1}{T}\begin{cases} 0 & D\text{ even} \\ \frac{(-q)^{\frac{D+1}{2}}l^D}{2^{\frac{D+1}{2}}!\pi^{\frac{D+1}{2}}} & D\text{ odd,} \end{cases}$$

these are smooth functions of the temperature T, for bounded $T \geq 0$, uniformly in the mass term q, for q in any closed interval of the positive real axis. When $q = 0$,

$$\log Z_{\mathcal{B}_D}(T, 0, l) = -\frac{\pi}{2l}\xi_D\left(-\frac{1}{2}; 0\right)\frac{1}{T} - \log\prod_{k\in(\mathbb{N}_0)^D}\left(1 - e^{-\frac{\pi}{lT}|k|}\right),$$

$$\log Z_{\mathcal{T}_D}(T, 0, l) = -\frac{\pi}{2l}\hat{\zeta}\left(-\frac{1}{2}; 0\right)\frac{1}{T} + \log T - \log\prod_{k\in\mathbb{Z}_0^D}\left(1 - e^{-\frac{\pi}{lT}|k|}\right) + \frac{1}{2}\log\rho,$$

where some values for the multidimensional Riemann zeta functions are given in 4.2 and 4.3 respectively. Using these expressions, we get the behavior of the main thermodynamic functions: partition function, energy, entropy, pressure of the radiation, and specific heat. For low T, fixed q and l, we get on the box

$$\log Z_{\mathcal{B}_D}(T, q, l) = \frac{1}{2}A_D(q, l)\frac{1}{T} +$$

$$+\frac{(-1)^{D+1}}{2^{D+2}}\log\rho\frac{1}{T}\sum_{i=0}^{\left[\frac{D-1}{2}\right]}\binom{D}{2i+1}\frac{(-1)^i}{(i+1)!}\frac{l^{2i+1}q^{i+1}}{\pi^i} + O(e^{-\frac{1}{T}}),$$

$$F_{\mathcal{B}_D}(T, q, l) = -\frac{1}{2}A_D(q, l) + \frac{(-1)^D}{2^{D+2}}\log\rho\sum_{i=0}^{\left[\frac{D-1}{2}\right]}\binom{D}{2i+1}\frac{(-1)^i}{(i+1)!}\frac{l^{2i+1}q^{i+1}}{\pi^i} + O(Te^{-\frac{1}{T}}),$$

$$S_{\mathcal{B}_D}(T, q, l) = O(T^{-1}e^{-\frac{1}{T}}), \qquad c_{\mathcal{B}_D}(T, q, l) = O(T^{-2}e^{-\frac{1}{T}}).$$

while on the torus, we must distinguish the $q = 0$ case:

$$\log Z_{\mathcal{T}_D}(T, q, l) = \frac{1}{2}B_D(q, l)\frac{1}{T} - \frac{1}{2}\log\rho\frac{1}{T}\begin{cases} 0 & D \text{ even} \\ \frac{(-q)^{\frac{D+1}{2}}l^D}{2^{\frac{D+1}{2}}!\pi^{\frac{D+1}{2}}} & D \text{ odd} \end{cases} + O(e^{-\frac{1}{T}}),$$

$$F_{\mathcal{T}_D}(T, q, l) = -\frac{1}{2}B_D(q, l) + \frac{1}{2}\log\rho\begin{cases} 0 & D \text{ even} \\ \frac{(-q)^{\frac{D+1}{2}}l^D}{2^{\frac{D+1}{2}}!\pi^{\frac{D+1}{2}}} & D \text{ odd} \end{cases} + O(Te^{-\frac{1}{T}}),$$

$$S_{\mathcal{T}_D}(T, q, l) = O(T^{-1}e^{-\frac{1}{T}}), \qquad c_{\mathcal{T}_D}(T, q, l) = O(T^{-2}e^{-\frac{1}{T}}),$$

$$\log Z_{\mathcal{T}_D}(T, 0, l) = -\frac{\pi}{2l}\hat{\zeta}\left(-\frac{1}{2}; 0\right)\frac{1}{T} + \log T + \frac{1}{2}\log\rho + O(e^{-\frac{1}{T}}),$$

$$F_{\mathcal{T}_D}(T, 0, l) = -T\log T + \hat{\zeta}\left(-\frac{1}{2}; 0\right)\frac{\pi}{2l} - \frac{1}{2}T\log\rho + O(Te^{-\frac{1}{T}}),$$

$$S_{\mathcal{T}_D}(T, 0, l) = \log T + \frac{1}{2}\log\rho + 1 + O(T^{-1}e^{-\frac{1}{T}}),$$

$$c_{\mathcal{T}_D}(T, 0, l) = 1 + O(T^{-2}e^{-\frac{1}{T}}).$$

Notice that the null mass case needs no independent treatment on the box. Recalling the remark at the end of 4.2 or the Proposition 5, we get the behaviors for high T, fixed q and l (cfr [16] or [23]),

$$\log Z_{\mathcal{B}_D}(T,q,l) = \frac{l^D}{\pi^{\frac{D+1}{2}}}\Gamma\left(\frac{D+1}{2}\right)\zeta_R(D+1)T^D + O(T^{D-1}),$$

$$\log Z_{\mathcal{T}_D}(T,q,l) = \frac{(2l)^D}{\pi^{\frac{D+1}{2}}}\Gamma\left(\frac{D+1}{2}\right)\zeta_R(D+1)T^D + O(T^{D-1}).$$

4.5. Critical volume

As anticipate in the introduction, we will analyze in this section the dependence on the volume of the pressure of the radiation at finite temperature. The analysis is performed for the two models described in section 4, that only differ for the boundary conditions: periodic or of Dirichlet type. For simplicity, just consider the zero mass case. By definition,

$$P(T,V) = \frac{\partial}{\partial V}T\log Z(T,q,l) = \frac{1}{2}\frac{\partial}{\partial V}(T\zeta'(0;2\pi T,0,l)) - \frac{1}{2}\frac{\partial}{\partial V}(\log\rho\ T\zeta(0;2\pi T,0,l)).$$

Notice that applying corollary 6 for the Dirichlet boundary conditions, we always have the vanishing of the zero mass zeta function at $s=0$, namely $\zeta(0;2\pi T,0,l)=0$ for the box. For periodic boundary condition, we get a non trivial term involving the renormalization constant ρ. Thus the analysis in the following holds for the torus only if we assume the renormalization constant to be volume independent ¶. With this assumption, we get at each fixed temperature

$$P_{\mathcal{T}_D}(T,V) = \frac{2\pi}{D}V^{-\frac{D+1}{D}}\left(\sum_{k\in\mathbb{Z}_0^D}\frac{|k|}{e^{\frac{2\pi|k|}{TV^{\frac{1}{D}}}} - 1} - H_D\right),$$

$$P_{\mathcal{B}_D}(T,V) = \frac{\pi}{D}V^{-\frac{D+1}{D}}\left(\sum_{k\in(\mathbb{N}_0)^D}\frac{|k|}{e^{\frac{\pi|k|}{TV^{\frac{1}{D}}}} - 1} - K_D\right),$$

where the constants are

$$H_D = -\frac{1}{2}\hat{\zeta}_D\left(-\frac{1}{2};0\right), \qquad K_D = -\frac{1}{2}\xi_D\left(-\frac{1}{2};0\right).$$

We show that for the periodic boundary condition there is, for all D, a value V_0 of the volume where the pressure changes sign, being attractive when $V < V_0$. We also show that the same happens for the Dirichlet boundary condition at the physical dimension $D=3$. For the torus, consider the function

$$g_D(x) = \sum_{k\in\mathbb{Z}_0^D}\frac{|k|}{e^{\frac{|k|}{x}} - 1} = \frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} + g_0(x),$$

¶ As observed in the introduction, we are not going to analyze here the renormalization aspects of the model. Beside, notice that even with a renormalization parameter depending on the volume, the volume effect still exists, but critical volume depends also on the explicit form of ρ. This one, as pointed out in [14], gives the connection between the model and the physical reality.

for $x > 0$ and the following inequality

Lemma 8 *If $x > ab$ and $y \geq b$ ($x, y, a, b > 0$), then*

$$e^{\frac{ab}{a+1}y}\left(e^{\frac{ab}{1+a}x} - 1\right) < \left(e^{xy} - 1\right).$$

Taking $b = 1$ and $a = \frac{1}{x_0}$, we get the following bounds for $g_D(x)$, when $x < x_0$,

$$\frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} < g_D(x) < \frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} + \frac{1}{e^{\frac{1}{1+x_0}\frac{1}{x}} - 1}C_D(x_0),$$

where $C_D(x_0) = \sum_{k \in \mathbb{Z}^D_{\{0,1\}}} |k| e^{-\frac{1}{1+x_0}|k|}$, is a positive constant. It is thus clear that the pressure changes sign for some value of V (and fixed T) if the constant H_D is positive. To show that this is the case, just use the reflection formula (lemma 5) that for the function $\hat{\zeta}_D(s; 0)$ takes the simpler form

$$\pi^{-s}\Gamma(s)\hat{\zeta}_D(s; 0) = \pi^{s - \frac{D}{2}}\Gamma\left(\frac{D}{2} - s\right)\hat{\zeta}_D\left(\frac{D}{2} - s; 0\right).$$

Using lemma 8 with $b = 2$ for the box, we get for the function

$$f_D(x) = \sum_{k \in (\mathbb{N}_0)^D} \frac{|k|}{e^{\frac{|k|}{x}} - 1} = \frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} + f_0(x),$$

($x > 0$) similar bounds when $x < \frac{x_0}{2}$:

$$\frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} < g_D(x) < \frac{\sqrt{D}}{e^{\frac{\sqrt{D}}{x}} - 1} + \frac{1}{e^{\frac{2}{1+x_0}\frac{1}{x}} - 1}L_D(x_0),$$

where $L_D(x_0) = \sum_{k \in (\mathbb{N}_0)^D - \{1\}} |k| e^{-\frac{2}{1+x_0}|k|}$, is a positive constant. Furthermore, using simple bounds for the norm, we get a bound for the constant $L_D(x_0)$

$$L_D(x_0) \leq 2^{\frac{D-1}{2}} \frac{e^{-\frac{4D}{1+x_0}}\left(2 - e^{-\frac{2}{1+x_0}}\right)^D}{\left(1 - e^{-\frac{2}{1+x_0}}\right)^{2D}}.$$

We can not prove that K_D is positive in general, but we can analyze explicitly the low dimensional cases. We find that K_D is negative for $D = 2$, but it is positive for $D = 1$ and 3. This indicates that, up to renormalization, the presence/absence of a boundary does not affect the Casimir effect in the physical dimension for the model under study. Eventually, using the above bounds, we can provide bounds for the solution $x = x_*$ of $f_D(x) = K_D$. For example, if $D = 3$, we get $K_3 = 0.0050075$, and with $x_0 = 1.5923$, $C_3(x_0) \leq 0.23433$, and $0.19684 < x_* < 0.29613$.

References

[1] M. Atiyah, R. Bott and V.K. Patodi, Inventiones math. 19 (1973) 279-330;

[2] M. Bordag (ed.), *The Casimir effect 50 years later*, Leipzig (1998) World Sci. Pub.;

[3] M. Bordag, E. Elizalde, B. Geyer and K. Kirsten, Commun. Math. Phys. 179, 215 (1996);

[4] M. Bordag, E. Elizalde and K. Kirsten, J. Math. Phys. 37, 895 (1996);

[5] P.T. Bateman and E. Grosswald, Acta Arith. 9 (1964) 365-373;

[6] S. Chowla and A. Selberg, J. Reine Angew. Math. 227 (1967) 86-110;

[7] J. Choi and J.R. Quine, Rocky Mount. Jour. Math. 26 (1996) 719-729;

[8] G. Cognola, K. Kirsten, L. Vanzo and S. Zerbini, Phys. Rev. D 10 (1994) 5307-5312;

[9] J.S. Dowker, hep-th/0408038 (2004);

[10] E. Elizalde, JMP 35 (1994) 6100-6122;

[11] E. Elizalde *Ten Physical application of spectral zeta functions*. Lecture notes in Physics m 35 (Springer Verlag, Berlin, 1995);

[12] E. Elizalde, M. Bordag and K. Kirsten, J. Phys. A31, (1998) 1743;

[13] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific (1994);

[14] E. Elizalde and A.C. Tort, Phys. Rev. D66, 045033 (2002);

[15] P.B. Gilkey, Studies in Adv. Math. CRC Press, 1995;

[16] S.W. Hawking, Comm. Math. Phys. 55 (1977) 133-148;

[17] L. Hormader, *The analysis of linear partial differential operator.IV*, GMW 275 (1994) Springer;

[18] G. Illies, CMP 220 (2001) 69-94;

[19] P. Li and S.-T. Yau, Proc. of Symp. in Pure Math. XXXVI AMS (1980) 205-240;

[20] M. Lesch, Math. Nachr. 194 (1998) 139-170;

[21] R.B. Melrose, Proc. of Symp. in Pure Math. XXXVI AMS (1980) 257-274;

[22] K.A. Milton, *The Casimir effect*, World Sci. Pub. (2001);

[23] P. Ramond, *Quantum field theory: A modern primer*, Addison-Wesley (1989);

[24] D.B. Ray and I.M. Singer, Adv. Math. 7 (1974) 145-210;

[25] M. Reed and B. Simon, *Methods of modern mathematical physics. IV*, (1978) Academic Press;

[26] S. Rosenberg, *The Laplacian on a Riemannian manifold*, LMS 31;

[27] C.L. Seigel, Ann. of Math. (1943) 143-172;

[28] M. Spreafico, Rocky Mount. J. Maths 33 (2003);

[29] M. Spreafico, *Zeta invariants for Dirichlet series*, submitted to Pacific J. Math. (2003);

[30] P.R. Taylor, Quat. J. Math. Oxford 11 (1940) 177-182;

[31] A.A. Terras, Trans. AMS (1973) 477-486;

[32] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag (1976).